

ZERO-SUM RISK-SENSITIVE STOCHASTIC GAMES FOR CONTINUOUS TIME MARKOV CHAINS

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ABSTRACT. We study infinite horizon discounted-cost and ergodic-cost risk-sensitive zero-sum stochastic games for controlled continuous time Markov chains on a countable state space. For the discounted-cost game we prove the existence of value and saddle-point equilibrium in the class of Markov strategies under nominal conditions. For the ergodic-cost game we prove the existence of values and saddle point equilibrium by studying the corresponding Hamilton-Jacobi-Isaacs equation under a certain Lyapunov condition.

Key words: Risk-sensitive cost, infinite horizon discounted cost, infinite horizon ergodic cost, HJI equation, value, saddle point equilibrium.

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1. INTRODUCTION

This paper is a sequel to [8] where the risk sensitive continuous time Markov decision process is studied on a countable state space. In this paper we extend the result of [8] to risk-sensitive zero sum stochastic games for continuous time control Markov chains on a countable state space. A zero-sum risk-sensitive differential game has been studied in [1] and the corresponding discrete time problem studied in [2]. As noted in [1] and [2], the zero-sum risk-sensitive stochastic dynamic game is relevant in worst-case scenarios, for example, in financial applications when a risk-averse investor is trying to minimize his long-term portfolio loss against the market which, by default, is antagonistic and hence the maximizer. As a result the minimizer chooses the risk-aversion parameter $\theta > 0$ and tries to minimize his expected risk-sensitive costs. Thus the risk-sensitive parameter is positive. If $\theta < 0$ then minimizer would be risk-seeking. The maximizer is not risk-seeking but simply antagonistic to the minimizer. Under certain conditions we establish value and saddle point strategies for both players.

The rest of the paper is structured as follows. Section 2 deals with the description of the problem. In Section 3, we prove the existence of value and saddle-point equilibrium in the class of Markov strategies for the discounted-cost risk-sensitive zero-sum game. The analysis of ergodic-cost risk-sensitive

zero-sum game is carried out in Section 4. The paper is concluded in Section 5 with some concluding remarks.

2. PROBLEM DESCRIPTION

Let $U_i, i = 1, 2$, be compact metric space and $V_i = \mathcal{P}(U_i)$, space of probability measure on U_i with Prohorov topology. Let

$$U := U_1 \times U_2 \text{ and } V := V_1 \times V_2.$$

Let $\bar{\pi}_{ij} : U \rightarrow [0, \infty)$ for $i \neq j$ and $\bar{\pi}_{ii} : U \rightarrow \mathbb{R}$ for $i \in S$. Define $\pi_{ij} : V \rightarrow \mathbb{R}$ as follows: for $v := (v_1, v_2) \in V$,

$$\pi_{ij}(v_1, v_2) = \int_{U_2} \int_{U_1} \bar{\pi}_{ij}(u_1, u_2) v_1(du_1) v_2(du_2) := \int_U \bar{\pi}_{ij}(u) v(du),$$

where $u := (u_1, u_2) \in U$.

We consider a continuous time controlled Markov chain $Y(\cdot)$ with state space $S = \{1, 2, \dots\}$ and controlled rate matrix $\Pi_{v_1, v_2} = (\pi_{ij}(v_1, v_2))$, given by the stochastic integral

$$(2.1) \quad dY(t) = \int_{\mathbb{R}} h(Y(t-), v_1(t), v_2(t), z) \wp(dz dt).$$

Here $\wp(dz dt)$ is a Poisson random measure with intensity $m(dz)dt$, where $m(dz)$ denote the Lebesgue measure on \mathbb{R} . The control process $v(\cdot) := (v_1(\cdot), v_2(\cdot))$ takes values in V , and $h : S \times V \times \mathbb{R} \rightarrow \mathbb{R}$ is defined as follows:

$$(2.2) \quad h(i, v, z) = \begin{cases} j - i & \text{if } z \in \Delta_{ij}(v) \\ 0 & \text{otherwise,} \end{cases}$$

where $v := (v_1, v_2)$ and $\{\Delta_{ij}(v) : i \neq j, i, j \in S\}$ denote intervals of the form $[a, b)$ with length of $\Delta_{ij}(v) = \pi_{ij}(v)$ which are pairwise disjoint for each fixed $v \in V$.

If $v_i(t) = \bar{v}_i(t, Y(t-))$ for some measurable map $\bar{v}_i : [0, \infty) \times S \rightarrow V_i$, then $v_i(\cdot)$ is called a Markov strategy for the i th player. With an abuse of notation the map \bar{v}_i itself is called a Markov strategy of player i . A Markov strategy $\bar{v}_i(\cdot)$ is called a stationary strategy if the map \bar{v}_i does not depend explicitly on time. We denote the set of all Markov strategies by \mathcal{M}_i and set of all stationary strategies by \mathcal{S}_i for the i th player.

Throughout this paper we assume that:

- $\bar{\pi}_{ij}(u) \geq 0$ for all $i \neq j$, $u \in U$ and the (infinite) matrix $(\bar{\pi}_{ij}(u))$ is conservative, i.e.,

$$\sum_{j \in S} \bar{\pi}_{ij}(u) = 0 \text{ for } i \in S \text{ and } u \in U.$$

- The function $\bar{\pi}_{ij}$ are continuous and

$$\sup_{i \in S, u \in U} [-\bar{\pi}_{ii}(u)] := M < \infty.$$

The existence of a unique weak solution to the equation (2.1) for a pair of Markov strategies (v_1, v_2) for a given initial distribution $\mu \in \mathcal{P}(S)$ follows using the above assumption, see [[6], Theorem 2.3, Theorem 2.5, pp.14-15].

Let $\bar{r} : S \times U_1 \times U_2 \rightarrow [0, \infty)$ be the running cost function. Throughout this paper, we assume that the function $\bar{r}(\cdot)$ is nonnegative, bounded and continuous.

We list the commonly used notations below.

- $C_b[a, b]$ denotes the set of all bounded and continuous functions on $[a, b]$.
- $B(S)$ denotes the set of all bounded functions on S .
- $C^1(a, b)$ denotes the set of all continuously differentiable functions on (a, b) .
- $C_c^\infty(a, b)$ denotes the set of all infinitely differentiable functions on (a, b) with compact support.
- $C_b([a, b] \times S)$ denotes the set of all functions $f : [a, b] \times S \rightarrow \mathbb{R}$ such that $f(t, i) \in C_b[a, b]$, for each $i \in S$.
- $C^1((a, b) \times S)$ denotes the set of all functions $f : (a, b) \times S \rightarrow \mathbb{R}$ such that $f(t, i) \in C^1(a, b)$, for each $i \in S$.

Set

$$B_W(S) = \{h : S \rightarrow \mathbb{R} \mid \sup_{i \in S} \frac{|h(i)|}{W(i)} < \infty\},$$

where W is the Lyapunov function as in (A1) (to be described in Section 4). Define for $h \in B_W(S)$,

$$\|h\|_W = \sup_{i \in S} \frac{|h(i)|}{W(i)}.$$

Then $B_W(S)$ is a Banach space with the norm $\|\cdot\|_W$.

2.1. Discounted Cost Criterion. For a pair of Markov strategies (v_1, v_2) , define α -discounted risk-sensitive cost by

$$(2.3) \quad \beta_\alpha^{v_1, v_2}(\theta, i) = \frac{1}{\theta} \ln E_i^{v_1, v_2} \left[e^{\theta \int_0^\infty e^{-\alpha t} r(Y(t), v_1(t, Y(t-)), v_2(t, Y(t-))) dt} \right]$$

for some $\theta \in (0, \Theta)$, and a fixed $\Theta > 0$, $\alpha > 0$ is the discount factor, $Y(\cdot)$ is the Markov chain corresponding to $(v_1, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2$ with $Y(0) = i$, and $r : S \times V \rightarrow \mathbb{R}_+$ is given by

$$r(i, v_1, v_2) = \int_{U_2} \int_{U_1} \bar{r}(i, u_1, u_2) v_1(du_1) v_2(du_2) := \int_U \bar{r}(i, u) v(du),$$

where $u := (u_1, u_2)$ and $v := (v_1, v_2)$.

Let $\theta \in (0, \Theta)$ be the “risk-sensitive parameter” chosen by the minimizer. When the state of the system is i and players 1,2, choose strategies $v_1 \in \mathcal{M}_1$, $v_2 \in \mathcal{M}_2$ respectively, the minimizer (player 1) tries to minimize his infinite-horizon discounted risk-sensitive cost $\beta_\alpha^{v_1, v_2}(\theta, i)$ over his strategies whereas

the maximizer (player 2) tries to maximize the same over his strategies. A strategy $v_1^* \in \mathcal{M}_1$ is called optimal for player 1 for $(\theta, i) \in (0, \Theta) \times S$, if

$$\beta_{\alpha}^{v_1^*, \tilde{v}_2}(\theta, i) \leq \sup_{v_2 \in \mathcal{M}_2} \inf_{v_1 \in \mathcal{M}_1} \beta_{\alpha}^{v_1, v_2}(\theta, i) := \underline{\beta}(\alpha, \theta, i) \text{ (lower value)}$$

for any $\tilde{v}_2 \in \mathcal{M}_2$. Similarly a strategy $v_2^* \in \mathcal{M}_2$ is called optimal for player 2 for $(\theta, i) \in (0, \Theta) \times S$, if

$$\beta_{\alpha}^{\tilde{v}_1, v_2^*}(\theta, i) \geq \inf_{v_1 \in \mathcal{M}_1} \sup_{v_2 \in \mathcal{M}_2} \beta_{\alpha}^{v_1, v_2}(\theta, i) := \overline{\beta}(\alpha, \theta, i) \text{ (upper value)}$$

for any $\tilde{v}_1 \in \mathcal{M}_1$. The game has value if

$$(2.4) \quad \underline{\beta}(\alpha, \theta, i) = \overline{\beta}(\alpha, \theta, i) = \beta(\alpha, \theta, i) \quad \forall i \in S, \forall \theta \in (0, \Theta).$$

A pair of strategies (v_1^*, v_2^*) at which this value is attained is called a saddle-point equilibrium, and then v_1^* is optimal for player 1, and v_2^* is optimal for player 2.

2.2. Ergodic Cost Criterion. For a pair of Markov strategies (v_1, v_2) , the risk-sensitive ergodic cost is given by

$$(2.5) \quad \rho^{v_1, v_2}(\theta, i) = \limsup_{T \rightarrow \infty} \frac{1}{\theta T} \ln E_i^{v_1, v_2} \left[e^{\theta \int_0^T r(Y(t), v_1(t, Y(t-)), v_2(t, Y(t-))) dt} \right],$$

for some $\theta \in (0, \Theta)$, and a fixed $\Theta > 0$, $Y(\cdot)$ is the Markov chain corresponding to $(v_1, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2$ with $Y(0) = i$.

Optimal strategies, saddle point equilibrium, etc. for this criterion are defined analogously. The ergodic cost ρ^{v_1, v_2} may depend on (θ, i) .

3. ANALYSIS OF DISCOUNTED COST CRITERION

We carry out our analysis of the discounted cost criterion via the criterion

$$(3.1) \quad \xi_{\alpha}^{v_1, v_2}(\theta, i) = E_i^{v_1, v_2} \left[e^{\theta \int_0^{\infty} e^{-\alpha t} r(Y(t), v_1(t, Y(t-)), v_2(t, Y(t-))) dt} \right].$$

Since logarithmic is an increasing function, therefore the optimal strategies for the criterion (2.3) are optimal strategies for the above criterion.

Corresponding to the cost criterion (3.1), the value function is defined as

$$\overline{\psi}_{\alpha}(\theta, i) = \inf_{v_1 \in \mathcal{M}_1} \sup_{v_2 \in \mathcal{M}_2} \xi_{\alpha}^{v_1, v_2}(\theta, i)$$

and

$$\underline{\psi}_{\alpha}(\theta, i) = \sup_{v_2 \in \mathcal{M}_2} \inf_{v_1 \in \mathcal{M}_1} \xi_{\alpha}^{v_1, v_2}(\theta, i).$$

Using dynamic programming heuristics, the Hamilton-Jacobi-Isaacs (HJI) equations for discounted cost criterion are given by

$$\begin{aligned} \alpha \theta \frac{d\psi_{\alpha}}{d\theta}(\theta, i) &= \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\Pi_{v_1, v_2} \psi_{\alpha}(\theta, i) + \theta r(i, v_1, v_2) \psi_{\alpha}(\theta, i) \right] \\ &= \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2} \psi_{\alpha}(\theta, i) + \theta r(i, v_1, v_2) \psi_{\alpha}(\theta, i) \right] \\ (3.2) \quad \psi_{\alpha}(0, i) &= 1, \end{aligned}$$

where $\Pi_{v_1, v_2} f(i) := \sum_{j \in S} \pi_{ij}(v_1, v_2) f(j)$, for any function $f(i)$.

Next we prove that the equations (3.2) have a smooth, bounded solution. Fix $\epsilon > 0$ and consider the ordinary differential equation (ODE)

$$\begin{aligned} \alpha \theta \frac{d\psi_\alpha^\epsilon}{d\theta}(\theta, i) &= \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\Pi_{v_1, v_2} \psi_\alpha^\epsilon(\theta, i) + \theta r(i, v_1, v_2) \psi_\alpha^\epsilon(\theta, i) \right] \\ (3.3) \quad &= \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2} \psi_\alpha^\epsilon(\theta, i) + \theta r(i, v_1, v_2) \psi_\alpha^\epsilon(\theta, i) \right] \end{aligned}$$

$$\psi_\alpha^\epsilon(\epsilon, i) = e^{\frac{\epsilon}{\alpha} \|r\|_\infty} := h_\epsilon,$$

where $\|\cdot\|_\infty$ denotes the supnorm. Note that the second equality follows from Fan's minimax theorem, see [[4], Theorem 3].

Let $\delta > 0$. Define the nonlinear operator $T : C_b([\epsilon, \epsilon + \delta] \times S) \rightarrow C_b([\epsilon, \epsilon + \delta] \times S)$ by

$$Tf(\eta, i) := e^{\frac{\epsilon}{\alpha} \|r\|_\infty} + \frac{1}{\alpha} \int_\epsilon^\eta \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\frac{1}{\theta} \Pi_{v_1, v_2} f(\theta, i) + r(i, v_1, v_2) f(\theta, i) \right] d\theta.$$

By using the fact $\sup_{i \in S, u \in U} [-\bar{\pi}_{ii}(u)] = M < \infty$ and r is bounded, we have

$$\|Tf_1 - Tf_2\|_\infty \leq \frac{1}{\alpha} \left[\|r\|_\infty \delta + 2M \ln \left(1 + \frac{\delta}{\epsilon} \right) \right] \|f_1 - f_2\|_\infty.$$

Choose δ such that $\frac{1}{\alpha} \left[\|r\|_\infty \delta + 2M \ln \left(1 + \frac{\delta}{\epsilon} \right) \right] < 1$. Then T is a contraction operator. Therefore by Banach's fixed point theorem there exists a function $\psi_\alpha^\epsilon \in C_b([\epsilon, \epsilon + \delta] \times S)$ such that

$$\psi_\alpha^\epsilon(\eta, i) = e^{\frac{\epsilon}{\alpha} \|r\|_\infty} + \frac{1}{\alpha} \int_\epsilon^\eta \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\frac{1}{\theta} \Pi_{v_1, v_2} \psi_\alpha^\epsilon(\theta, i) + r(i, v_1, v_2) \psi_\alpha^\epsilon(\theta, i) \right] d\theta.$$

Note that the bracketed term in the above integrand is bounded and jointly continuous in (θ, v_1, v_2) . Since V_1 and V_2 are compact metric spaces, it follows that the integrand above is bounded and continuous in $\theta \in [\epsilon, \epsilon + \delta]$. Thus it follows that ψ_α^ϵ is in $C^1((\epsilon, \epsilon + \delta] \times S) \cap C_b([\epsilon, \epsilon + \delta] \times S)$. Proceeding in this way we get a $C^1((\epsilon, \Theta) \times S) \cap C_b([\epsilon, \Theta] \times S)$ solution for the ODE (3.3). Let

$$\bar{v}_i : (0, \Theta) \times S \rightarrow V_i, \quad i = 1, 2,$$

be measurable functions such that

$$\begin{aligned} &\inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\Pi_{v_1, v_2} \psi_\alpha^\epsilon(\theta, i) + \theta r(i, v_1, v_2) \psi_\alpha^\epsilon(\theta, i) \right] \\ (3.4) \quad &= \sup_{v_2 \in V_2} \left[\Pi_{\bar{v}_1, v_2} \psi_\alpha^\epsilon(\theta, i) + \theta r(i, \bar{v}_1(\theta, i), v_2) \psi_\alpha^\epsilon(\theta, i) \right] \end{aligned}$$

and

$$(3.5) \quad \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2} \psi_\alpha^\epsilon(\theta, i) + \theta r(i, v_1, v_2) \psi_\alpha^\epsilon(\theta, i) \right] \\ = \inf_{v_1 \in V_1} \left[\Pi_{v_1, \bar{v}_2} \psi_\alpha^\epsilon(\theta, i) + \theta r(i, v_1, \bar{v}_2(\theta, i)) \psi_\alpha^\epsilon(\theta, i) \right].$$

The existence of such measurable maps are ensured by Beneš measurable selection theorem, see [3]. Let

$$v_i^* : \mathbb{R}_+ \times S \rightarrow V_i, \quad i = 1, 2,$$

be defined by

$$v_i^*(t, i) = \bar{v}_i(\theta e^{-\alpha t}, i), \quad i = 1, 2.$$

Set $\theta(t) = \theta e^{-\alpha t}$ and define T_ϵ by

$$T_\epsilon = \inf\{t \geq 0 : \theta(t) = \epsilon\}.$$

For $(v_1^*, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2$, applying Itô formula (see [6], Appendix C, pp. 218-219) to the function

$$e^{\int_0^t \theta(s) r(Y(s), v_1^*(s, Y(s-)), v_2(s, Y(s-))) ds} \psi_\alpha^\epsilon(\theta(t), Y(t)),$$

we obtain

$$E_i^{v_1, v_2} [e^{\int_0^{T_\epsilon} \theta(s) r(Y(s), v_1^*(s, Y(s-)), v_2(s, Y(s-))) ds} \psi_\alpha^\epsilon(\theta(T_\epsilon), Y(T_\epsilon))] - \psi_\alpha^\epsilon(\theta, i) \\ = E_i^{v_1, v_2} \left[\int_0^{T_\epsilon} e^{\int_0^t \theta(s) r(Y(s), v_1^*(s, Y(s-)), v_2(s, Y(s-))) ds} \left\{ -\alpha \theta(t) \frac{d\psi_\alpha^\epsilon}{d\theta}(\theta(t), Y(t)) \right. \right. \\ \left. \left. + \Pi_{v_1, v_2} \psi_\alpha^\epsilon(\theta(t), Y(t)) + \theta(t) r(Y(t), v_1^*(t, Y(t-)), v_2(t, Y(t-))) \psi_\alpha^\epsilon(\theta(t), Y(t)) \right\} dt \right].$$

Since ψ_α^ϵ satisfies (3.4), we obtain

$$E_i^{v_1^*, v_2} [e^{\int_0^{T_\epsilon} \theta(s) r(Y(s), v_1^*(s, Y(s-)), v_2(s, Y(s-))) ds} h_\epsilon] - \psi_\alpha^\epsilon(\theta, i) \leq 0,$$

where h_ϵ is as in (3.3). Since v_2 is arbitrary, we get

$$(3.6) \quad \psi_\alpha^\epsilon(\theta, i) \geq \sup_{v_2 \in \mathcal{M}_2} E_i^{v_1^*, v_2} \left[h_\epsilon e^{\int_0^{T_\epsilon} \theta(s) r(Y(s), v_1^*(s, Y(s-)), v_2(s, Y(s-))) ds} \right].$$

Using analogous arguments, we can show that

$$(3.7) \quad \psi_\alpha^\epsilon(\theta, i) \leq \inf_{v_1 \in \mathcal{M}_1} E_i^{v_1, v_2^*} \left[h_\epsilon e^{\int_0^{T_\epsilon} \theta(s) r(Y(s), v_1(s, Y(s-)), v_2^*(s, Y(s-))) ds} \right].$$

Therefore, from (3.6) and (3.7), we obtain

$$(3.8) \quad \psi_\alpha^\epsilon(\theta, i) = \sup_{v_2 \in \mathcal{M}_2} \inf_{v_1 \in \mathcal{M}_1} E_i^{v_1, v_2} \left[h_\epsilon e^{\int_0^{T_\epsilon} \theta(s) r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right] \\ = \inf_{v_1 \in \mathcal{M}_1} \sup_{v_2 \in \mathcal{M}_2} E_i^{v_1, v_2} \left[h_\epsilon e^{\int_0^{T_\epsilon} \theta(s) r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right].$$

Next we take limit of ψ_α^ϵ as $\epsilon \rightarrow 0$ and prove that the limit function satisfies (3.2), i.e., we prove the following theorem.

Theorem 3.1. *There exists a unique solution ψ_α in the class $C_b((0, \Theta) \times S) \cap C^1((0, \Theta) \times S)$ to (3.2). The solution admits the following representation*

$$\begin{aligned}\psi_\alpha(\theta, i) &= \sup_{v_2 \in \mathcal{M}_2} \inf_{v_1 \in \mathcal{M}_1} E_i^{v_1, v_2} \left[e^{\theta \int_0^\infty e^{-\alpha s} r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right] \\ &= \inf_{v_1 \in \mathcal{M}_1} \sup_{v_2 \in \mathcal{M}_2} E_i^{v_1, v_2} \left[e^{\theta \int_0^\infty e^{-\alpha s} r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right].\end{aligned}$$

Furthermore ψ_α is the value function for the discounted cost criterion (3.1). Moreover, a saddle point equilibrium exists in $\mathcal{M}_1 \times \mathcal{M}_2$.

Proof. First recall the stochastic representation of ψ_α^ϵ from (3.8),

$$\begin{aligned}\psi_\alpha^\epsilon(\theta, i) &= \sup_{v_2 \in \mathcal{M}_2} \inf_{v_1 \in \mathcal{M}_1} E_i^{v_1, v_2} \left[h_\epsilon e^{\int_0^{T_\epsilon} \theta(s) r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right] \\ &= \inf_{v_1 \in \mathcal{M}_1} \sup_{v_2 \in \mathcal{M}_2} E_i^{v_1, v_2} \left[h_\epsilon e^{\int_0^{T_\epsilon} \theta(s) r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right].\end{aligned}$$

From the representation of ψ_α^ϵ , we have

$$1 \leq \psi_\alpha^\epsilon(\theta, i) \leq h_\epsilon e^{\frac{\theta}{\alpha} \|r\|_\infty (1 - e^{-\alpha T_\epsilon})} = e^{\frac{\theta}{\alpha} \|r\|_\infty}$$

for every $\epsilon > 0$, and all (θ, i) .

By closely mimicking the arguments in the proof of [[5], Theorem 3.4], it follows that the HJB equation (3.2) has a solution in $C_b((0, \Theta) \times S) \cap C^1((0, \Theta) \times S)$. Let

$$\bar{v}_i : (0, \Theta) \times S \rightarrow V_i, \quad i = 1, 2,$$

be measurable selectors such that

$$\begin{aligned}(3.9) \quad & \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\Pi_{v_1, v_2} \psi_\alpha(\theta, i) + \theta r(i, v_1, v_2) \psi_\alpha(\theta, i) \right] \\ &= \sup_{v_2 \in V_2} \left[\Pi_{\bar{v}_1, v_2} \psi_\alpha(\theta, i) + \theta r(i, \bar{v}_1(\theta, i), v_2) \psi_\alpha(\theta, i) \right]\end{aligned}$$

and

$$\begin{aligned}(3.10) \quad & \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2} \psi_\alpha(\theta, i) + \theta r(i, v_1, v_2) \psi_\alpha(\theta, i) \right] \\ &= \inf_{v_1 \in V_1} \left[\Pi_{v_1, \bar{v}_2} \psi_\alpha(\theta, i) + \theta r(i, v_1, \bar{v}_2(\theta, i)) \psi_\alpha(\theta, i) \right].\end{aligned}$$

Let

$$v_i^* : \mathbb{R}_+ \times S \rightarrow V_i, \quad i = 1, 2,$$

be defined by

$$(3.11) \quad v_i^*(t, i) = \bar{v}_i(\theta e^{-\alpha t}, i), \quad i = 1, 2.$$

For $(v_1^*, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2$, applying Itô formula to the function

$$e^{\int_0^t \theta(s) r(Y(s), v_1^*(s, Y(s-)), v_2(s, Y(s-))) ds} \psi_\alpha(\theta(t), Y(t))$$

and using (3.9), we get

$$E_i^{v_1^*, v_2} [e^{\int_0^T \theta(s) r(Y(s), v_1^*(s, Y(s-)), v_2(s, Y(s-))) ds} \psi_\alpha(\theta(T), Y(T))] - \psi_\alpha(\theta, i) \leq 0.$$

Since v_2 is arbitrary, we get

$$\psi_\alpha(\theta, i) \geq \sup_{v_2 \in \mathcal{M}_2} E_i^{v_1^*, v_2} \left[\psi_\alpha(\theta(T), Y(T)) e^{\int_0^T \theta(s) r(Y(s), v_1^*(s, Y(s-)), v_2(s, Y(s-))) ds} \right].$$

Since $1 \leq \bar{\psi}_\alpha^\epsilon \leq e^{\frac{\theta}{\alpha} \|r\|_\infty}$ for all $\epsilon > 0$, we get

$$\psi_\alpha(\theta, i) \geq \sup_{v_2 \in \mathcal{M}_2} E_i^{v_1^*, v_2} \left[e^{\int_0^T \theta(s) r(Y(s), v_1^*(s, Y(s-)), v_2(s, Y(s-))) ds} \right].$$

By using monotone convergence theorem for letting $T \rightarrow \infty$ in the above we obtain

$$(3.12) \quad \psi_\alpha(\theta, i) \geq \sup_{v_2 \in \mathcal{M}_2} E_i^{v_1^*, v_2} \left[e^{\int_0^\infty \theta(s) r(Y(s), v_1^*(s, Y(s-)), v_2(s, Y(s-))) ds} \right].$$

Using analogous arguments we can show that

$$(3.13) \quad \psi_\alpha(\theta, i) \leq \inf_{v_1 \in \mathcal{M}_1} E_i^{v_1, v_2^*} \left[e^{\int_0^\infty \theta(s) r(Y(s), v_1(s, Y(s-)), v_2^*(s, Y(s-))) ds} \right].$$

Therefore, from (3.12) and (3.13), we obtain

$$\begin{aligned} \psi_\alpha(\theta, i) &= \sup_{v_2 \in \mathcal{M}_2} \inf_{v_1 \in \mathcal{M}_1} E_i^{v_1, v_2} \left[e^{\int_0^\infty \theta(s) r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right] \\ &= \inf_{v_1 \in \mathcal{M}_1} \sup_{v_2 \in \mathcal{M}_2} E_i^{v_1, v_2} \left[e^{\int_0^\infty \theta(s) r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right]. \end{aligned}$$

It is easy to check that ψ_α is the value function for the discounted cost criterion (3.1). Moreover, the pair of Markov strategies given by (3.11) forms a saddle point equilibrium. This completes the proof. \square

4. ANALYSIS OF ERGODIC COST CRITERION

In this section we prove the existence of value and stationary Markov saddle point strategies for the ergodic cost criterion under the following assumption:

(A1) (Lyapunov condition) There exist constants $b > 0$, $\delta > 0$, a finite set C and a map $W : S \rightarrow [1, \infty)$ with $W(i) \rightarrow \infty$ as $i \rightarrow \infty$, such that

$$\Pi_v W(i) \leq -2\delta W(i) + bI_C(i), \quad i \in S, \quad v \in V.$$

Throughout this section, we assume that for every pair of stationary Markov strategies (v_1, v_2) the corresponding Markov chain is irreducible.

We carry out our analysis of the ergodic cost criterion as a limit of the corresponding finite horizon cost criterion given by

$$(4.1) \quad I_T(i, v_1, v_2) := E_i^{v_1, v_2} \left[e^{\int_0^T r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right]$$

where $Y(\cdot)$ is the Markov chain corresponding to $(v_1, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2$ with initial condition $i \in S$. Using the dynamic programming heuristics, the HJI

equations for the above cost criterion, are given by

$$\begin{aligned}
-\frac{d\phi}{dt}(t, i) &= \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\Pi_{v_1, v_2} \phi(t, i) + r(i, v_1, v_2) \phi(t, i) \right] \\
&= \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2} \phi(t, i) + r(i, v_1, v_2) \phi(t, i) \right] \\
(4.2) \quad \phi(T, i) &= 1.
\end{aligned}$$

As before, we can show the existence of a $C^1((0, T) \times S) \cap C_b([0, T] \times S)$ solution for the ODE (4.2). Using a standard application of Itô's formula we get

$$\begin{aligned}
\phi(t, i) &= \sup_{v_2 \in \mathcal{M}_2} \inf_{v_1 \in \mathcal{M}_1} E_i^{v_1, v_2} \left[e^{\int_t^T r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right] \\
&= \inf_{v_1 \in \mathcal{M}_1} \sup_{v_2 \in \mathcal{M}_2} E_i^{v_1, v_2} \left[e^{\int_t^T r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right].
\end{aligned}$$

Set $\psi(t, i) = \phi(T - t, i)$. Then ψ is the unique $C^1((0, T) \times S) \cap C_b([0, T] \times S)$ solution to

$$\begin{aligned}
\frac{d\psi}{dt}(t, i) &= \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\Pi_{v_1, v_2} \psi(t, i) + r(i, v_1, v_2) \psi(t, i) \right] \\
&= \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2} \psi(t, i) + r(i, v_1, v_2) \psi(t, i) \right] \\
\psi(0, i) &= 1.
\end{aligned}$$

Using Itô's formula, we obtain

$$\begin{aligned}
\psi(t, i) &= \sup_{v_2 \in \mathcal{M}_2} \inf_{v_1 \in \mathcal{M}_1} E_i^{v_1, v_2} \left[e^{\int_0^t r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right] \\
&= \inf_{v_1 \in \mathcal{M}_1} \sup_{v_2 \in \mathcal{M}_2} E_i^{v_1, v_2} \left[e^{\int_0^t r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right].
\end{aligned}$$

Formally, using separation of variables, we write

$$\psi(t, i) = e^{\rho t} \hat{\psi}(i).$$

This yields

$$\begin{aligned}
\rho \hat{\psi}(i) &= \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\Pi_{v_1, v_2} \hat{\psi}(i) + r(i, v_1, v_2) \hat{\psi}(i) \right] \\
(4.3) \quad &= \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2} \hat{\psi}(i) + r(i, v_1, v_2) \hat{\psi}(i) \right].
\end{aligned}$$

The above equation is the HJI equation for the ergodic cost (2.5).

We now proceed to make a rigorous analysis of the above. First we truncate our cost function which plays a crucial role to derive the HJI equations and find the value of the game. Let $r_n : S \times V \rightarrow [0, \infty)$ be given by

$$(4.4) \quad r_n := \begin{cases} r & \text{if } i \in \{1, 2, \dots, n\} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned}\psi^n(t, i) &= \sup_{v_2 \in \mathcal{M}_2} \inf_{v_1 \in \mathcal{M}_1} E_i^{v_1, v_2} \left[e^{\int_0^t r_n(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right] \\ &= \inf_{v_1 \in \mathcal{M}_1} \sup_{v_2 \in \mathcal{M}_2} E_i^{v_1, v_2} \left[e^{\int_0^t r_n(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right].\end{aligned}$$

Then, as above, we can show that ψ^n is the unique solution in $C^1((0, T) \times S) \cap C_b([0, T] \times S)$ to

$$\begin{aligned}\frac{d\psi^n}{dt}(t, i) &= \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\Pi_{v_1, v_2} \psi^n(t, i) + r_n(i, v_1, v_2) \psi^n(t, i) \right] \\ &= \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2} \psi^n(t, i) + r_n(i, v_1, v_2) \psi^n(t, i) \right] \\ \psi^n(0, i) &= 1.\end{aligned}$$

Now onward, we fix a reference state $i_0 \in S$ such that $W(i_0) \geq 1 + \frac{b}{\delta}$ and set

$$(4.5) \quad \bar{\psi}^n(t, i) = \frac{\psi^n(t, i)}{\psi^n(t, i_0)}.$$

Then it is easy to see that $\bar{\psi}^n$ is the unique solution in $C^1((0, T) \times S) \cap C_b([0, T] \times S)$ to

$$\begin{aligned}\frac{d\bar{\psi}^n}{dt}(t, i) + \frac{\bar{\psi}^n(t, i)}{\psi^n(t, i_0)} \frac{d\psi^n}{dt}(t, i_0) &= \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\Pi_{v_1, v_2} \bar{\psi}^n(t, i) + r_n(i, v_1, v_2) \bar{\psi}^n(t, i) \right] \\ &= \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2} \bar{\psi}^n(t, i) + r_n(i, v_1, v_2) \bar{\psi}^n(t, i) \right] \\ (4.6) \quad \bar{\psi}^n(0, i) &= 1.\end{aligned}$$

Next we take limit as $t \rightarrow \infty$ in (4.6), to derive the existence of a solution for ergodic HJI equation with cost function r_n . For this we want to show that $\psi^n(t, i)$ is uniformly bounded (for each fixed n). To this end fix a strategy of player 2 and consider the corresponding optimal control problem for player 1.

Let $v_{2n}^* : \mathbb{R}_+ \times S \rightarrow V_2$ be a measurable map such that

$$\begin{aligned}&\inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\Pi_{v_1, v_2} \psi^n(t, i) + r_n(i, v_1, v_2) \psi^n(t, i) \right] \\ &= \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_{2n}^*(t, i)} \psi^n(t, i) + r_n(i, v_1, v_{2n}^*(t, i)) \psi^n(t, i) \right].\end{aligned}$$

We suppress the dependence of n on v_{2n}^* and write v_2^* instead.

For the fixed Markov strategy $v_2^* \in \mathcal{M}_2$ consider the pure jump processes given by

$$(4.7) \quad dY_{v_2^*}(t) = \int_{\mathbb{R}} h(Y_{v_2^*}(t-), v_1(t), v_2^*(t, Y_{v_2^*}(t-)), z) \varphi(dz dt),$$

where h is as in (2.2).

Now we consider a new auxiliary continuous time Markov decision problem

(CTMDP) corresponding to the process (4.7), i.e., player 2 fixes the strategy v_2^* and player 1 treats it as a CTMDP. First we define the set of all admissible controls denoted by \mathcal{A} .

A V_1 -valued process $v_1(\cdot)$ is said to be admissible if it is predictable and the equation

$$(4.8) \quad dY_{v_2^*}(t) = \int_{\mathbb{R}} h(Y_{v_2^*}(t-), v_1(t), v_2^*(t, Y_{v_2^*}(t-)), z) \varphi(dz dt)$$

has a unique weak solution for each initial Y_0 independent of $\varphi(dz dt)$.

For an admissible control $v_1(\cdot) \in \mathcal{A}$, the risk-sensitive cost for the finite horizon $[0, T]$ is defined by

$$E_{v_2^*}^n(i, v_1(\cdot)) = E_i^{v_1, v_2^*} \left[e^{\int_0^T r_n(Y_{v_2^*}(t), v_1(t), v_2^*(t, Y_{v_2^*}(t-))) dt} \right],$$

where $Y_{v_2^*}(\cdot)$ is the pure jump process (4.8) corresponding to $v_1(\cdot)$ and initial condition $i \in S$.

Consider the ODE

$$(4.9) \quad \begin{aligned} -\frac{d\Psi_{v_2^*}^n(t, i)}{dt} &= \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2^*}(t, i) \Psi_{v_2^*}^n(t, i) + r_n(i, v_1, v_2^*(t, i)) \Psi_{v_2^*}^n(t, i) \right] \\ \Psi_{v_2^*}^n(T, i) &= 1. \end{aligned}$$

Define the nonlinear operator $\mathcal{T} : C_b([0, T] \times S) \rightarrow C_b([0, T] \times S)$ by

$$\mathcal{T}f(t, i) := 1 + \int_t^T \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2^*}(s, i) f(s, i) + r_n(i, v_1, v_2^*(s, i)) f(s, i) \right] ds.$$

As before, we get the existence of a solution to (4.9) in $C_b([0, T] \times S)$. Using a standard application of Itô's formula, we obtain

$$\Psi_{v_2^*}^n(t, i) = \inf_{v_1(\cdot) \in \mathcal{A}} E_i^{v_1, v_2^*} \left[e^{\int_t^T r_n(Y_{v_2^*}(s), v_1(s), v_2^*(s, Y_{v_2^*}(s-))) ds} \right].$$

Set $\psi_{v_2^*}^n(t, i) = \Psi_{v_2^*}^n(T-t, i)$. Then $\psi_{v_2^*}^n$ is the unique solution in $C_b([0, T] \times S)$ to

$$(4.10) \quad \begin{aligned} \frac{d\psi_{v_2^*}^n(t, i)}{dt} &= \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2^*}(t, i) \psi_{v_2^*}^n(t, i) + r_n(i, v_1, v_2^*(t, i)) \psi_{v_2^*}^n(t, i) \right] \\ \psi_{v_2^*}^n(0, i) &= 1. \end{aligned}$$

Using Itô's formula, we obtain

$$\psi_{v_2^*}^n(t, i) = \inf_{v_1(\cdot) \in \mathcal{A}} E_i^{v_1, v_2^*} \left[e^{\int_0^t r_n(Y_{v_2^*}(s), v_1(s), v_2^*(s, Y_{v_2^*}(s-))) ds} \right].$$

It is easy to see that any minimizing selector in (4.10) corresponding to $\psi_{v_2^*}^n$ is optimal for the finite horizon CTMDP for player 1. Since any minimizing selector corresponds to a Markov control, we have

$$\psi_{v_2^*}^n(t, i) = \inf_{v_1 \in \mathcal{M}_1} E_i^{v_1, v_2^*} \left[e^{\int_0^t r_n(Y_{v_2^*}(s), v_1(s), v_2^*(s, Y_{v_2^*}(s-))) ds} \right].$$

For the reference state $i_0 \in S$, we define

$$\bar{\psi}_{v_2^*}^n(t, i) = \frac{\psi_{v_2^*}^n(t, i)}{\psi_{v_2^*}^n(t, i_0)}.$$

A simple calculation shows that $\bar{\psi}_{v_2^*}^n = \bar{\psi}^n$, where $\bar{\psi}^n$ is as in (4.5).

By closely mimicking the arguments in [[7], Theorem 3.1], one can easily get the following multiplicative DPP; we omit the details.

Theorem 4.1. *For any set $\tilde{S} \subseteq S$,*

$$\psi_{v_2^*}^n(t, i) = \inf_{v_1(\cdot) \in \mathcal{A}} E_i^{v_1, v_2^*} \left[e^{\int_0^{t \wedge \tau} r_n(Y_{v_2^*}(s), v_1(s), v_2^*(s), Y_{v_2^*}(s-)) ds} \psi_{v_2^*}^n(t - (t \wedge \tau), Y_{v_2^*}(t \wedge \tau)) \right], \quad t \geq 0,$$

where τ is the hitting time of the process $Y_{v_2^*}(\cdot)$ to the set \tilde{S} .

Using the similar arguments as in the proofs of [8], we can prove the following results; we omit the details.

Lemma 4.1. *Assume (A1). Let $Y(\cdot)$ be a process (2.1) corresponding to $(v_1, v_2^*) \in \mathcal{M}_1 \times \mathcal{M}_2$. Then*

$$E_i^{v_1, v_2^*} \left[e^{\delta \tau_{i_0}} \right] \leq W(i), \quad i \in S,$$

where $\tau_{i_0} = \inf\{t \geq 0 : Y(t) = i_0\}$.

Lemma 4.2. *Assume (A1) and $\|r\|_\infty \leq \delta$, where $\delta > 0$ is given in (A1). Then*

$$|\bar{\psi}_{v_2^*}^n(t, i)| \leq W(i), \quad t \geq 0, i \in S.$$

Lemma 4.3. *Assume (A1) and $\|r\|_\infty \leq \delta$, where $\delta > 0$ is given in (A1). Then*

$$\sup_{t > 0, i \in S} \|\bar{\psi}_{v_2^*}^n(t, i)\|_\infty < \infty.$$

Proof. Let $i \geq n + 1$ and $Y_{v_2^*}(\cdot)$ be the solution corresponding to $\hat{v}(\cdot) \in \mathcal{M}_1$ with initial condition i . Then from Theorem 4.1, we have

$$\begin{aligned} \psi_{v_2^*}^n(t, i) &\leq E_i^{\hat{v}, v_2^*} \left[e^{\int_0^{t \wedge \tau} r_n(Y_{v_2^*}(s), \hat{v}(s), Y_{v_2^*}(s), v_2^*(s), Y_{v_2^*}(s-)) ds} \psi_{v_2^*}^n(t - (t \wedge \tau), Y_{v_2^*}(t \wedge \tau)) \right], \\ &= E_i^{\hat{v}, v_2^*} \left[\psi_{v_2^*}^n(t - (t \wedge \tau), Y_{v_2^*}(t \wedge \tau)) I\{t \leq \tau\} \right] \\ &\quad + E_i^{\hat{v}, v_2^*} \left[\psi_{v_2^*}^n(t - (t \wedge \tau), Y_{v_2^*}(t \wedge \tau)) I\{t > \tau\} \right] \\ &= E_i^{\hat{v}, v_2^*} \left[\psi_{v_2^*}^n(0, Y_{v_2^*}(t)) I\{t \leq \tau\} \right] \\ &\quad + E_i^{\hat{v}, v_2^*} \left[\psi_{v_2^*}^n(t - \tau, Y_{v_2^*}(\tau)) I\{t > \tau\} \right] \\ &\leq 1 + E_i^{\hat{v}, v_2^*} \left[\psi_{v_2^*}^n(t, Y_{v_2^*}(\tau)) I\{t > \tau\} \right], \end{aligned}$$

where

$$\tau = \inf\{t \geq 0 : Y_{v_2^*}(t) \in \{1, 2, \dots, n\}\}.$$

In the last inequality we used the fact that $\psi_{v_2^*}^n(\cdot, i)$ is nondecreasing in t for each fixed i . Hence

$$\bar{\psi}_{v_2^*}^n(t, i) \leq 1 + \max_{j=1, \dots, n} \bar{\psi}_{v_2^*}^n(t, j) \leq 1 + \max_{j=1, \dots, n} W(j)$$

since $\psi_{v_2^*}^n(t, i_0) \geq 1$ and last inequality follows from Lemma 4.2. Therefore for each $n \geq 1$, $\bar{\psi}_{v_2^*}^n$ is bounded. \square

Remark 4.1. From Lemma 4.3, it follows that for each n , $\bar{\psi}_{v_2^*}^n$ is uniformly bounded (in t and i) and that bound is independent of the v_2^* . Therefore, we conclude that for each n , $\bar{\psi}^n$ is also uniformly bounded (in t and i), since $\bar{\psi}_{v_2^*}^n = \bar{\psi}^n$.

Lemma 4.4. Assume (A1) and $\|r\|_\infty \leq \delta$, where $\delta > 0$ is given in (A1). Then

$$\sup_{t \geq 0} \left\| \frac{1}{\psi^n(t, i_0)} \frac{d\psi^n}{dt}(t, \cdot) \right\|_W < \infty.$$

Proof. Note that

$$\begin{aligned} \frac{1}{\psi^n(t, i_0)} \frac{d\psi^n}{dt}(t, i) &= \frac{d\bar{\psi}^n}{dt}(t, i) + \frac{\bar{\psi}^n(t, i)}{\psi^n(t, i_0)} \frac{d\psi^n}{dt}(t, i_0) \\ &= \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\Pi_{v_1, v_2} \bar{\psi}^n(t, i) + r_n(i, v_1, v_2) \bar{\psi}^n(t, i) \right] \\ &= \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2} \bar{\psi}^n(t, i) + r_n(i, v_1, v_2) \bar{\psi}^n(t, i) \right]. \end{aligned}$$

The second equality follows from (4.6). Now the result follows from the fact

$$\sup_{i \in S, u \in U} [-\bar{\pi}_{ii}(u)] = M < \infty$$

and Remark 4.1. \square

Now we prove the existence of a solution to the HJI equation for the cost function r_n .

Theorem 4.2. Assume (A1) and $\|r\|_\infty \leq \delta$, where $\delta > 0$ is given in (A1). Then the equation

$$\begin{aligned} \rho^n \hat{\psi}^n(i) &= \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\Pi_{v_1, v_2} \hat{\psi}^n(i) + r_n(i, v_1, v_2) \hat{\psi}^n(i) \right] \\ (4.11) \quad &= \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2} \hat{\psi}^n(i) + r_n(i, v_1, v_2) \hat{\psi}^n(i) \right] \end{aligned}$$

has a solution $(\rho^n, \hat{\psi}^n(i))$ satisfying $\hat{\psi}^n(i_0) = 1$. Also

$$(4.12) \quad \rho^n \leq \sup_{v_2 \in \mathcal{M}_2} \inf_{v_1 \in \mathcal{M}_1} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln E_i^{v_1, v_2} \left[e^{\int_0^T r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right]$$

and

$$(4.13) \quad 0 < \hat{\psi}^n(i) \leq W(i), \quad n \geq 1, i \in S.$$

Proof. Using mean value theorem and Remark 4.1, there exists $s(t, i) \in [t, 2t]$, $t > 0$ such that

$$\lim_{t \rightarrow \infty} \frac{d\bar{\psi}^n}{dt}(s(t, i), i) = 0.$$

By Lemma 4.2, we have

$$\sup_{t \geq 0} |\bar{\psi}^n(s(t, i), i)| \leq W(i).$$

Using a diagonalization argument, along a subsequence, $\bar{\psi}^n(s(t, i), i) \rightarrow \hat{\psi}^n(i)$, for each $i \in S$ for some $\hat{\psi}^n \in B_W(S)$.

By Lemma 4.4, we have

$$\sup_{t \geq 0} \left| \frac{1}{\psi^n(s(t, i), i_0)} \frac{d\psi^n}{dt}(s(t, i), i_0) \right| < \infty.$$

Therefore, along a further subsequence denoted by the same notation by an abuse of notation, we have

$$\frac{1}{\psi^n(s(t, i), i_0)} \frac{d\psi^n}{dt}(s(t, i), i_0) \rightarrow \rho^n, \text{ for some } \rho^n \in \mathbb{R}.$$

Hence, along a suitable subsequence, we have

$$\frac{d\bar{\psi}^n}{dt}(s(t, i), i) + \frac{\bar{\psi}^n(s(t, i), i)}{\psi^n(s(t, i), i_0)} \frac{d\psi^n}{dt}(s(t, i), i_0) \rightarrow \hat{\psi}^n(i) \rho^n.$$

By letting $t \rightarrow \infty$ in (4.6) at $t = s(t, i)$ along a suitable subsequence, and using (A1) it follows that $(\rho^n, \hat{\psi}^n(i))$ is a solution to the equation (4.11) satisfying $\hat{\psi}^n(i_0) = 1$.

Let $v_n^* = (v_{1n}^*, v_{2n}^*) : S \rightarrow V$ be a min-max selector such that

$$\begin{aligned} & \sup_{v_2 \in V_2} \left[\Pi_{v_{1n}^*(i), v_2} \hat{\psi}^n(i) + r_n(i, v_{1n}^*(i), v_2) \hat{\psi}^n(i) \right] \\ &= \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \left[\Pi_{v_1, v_2} \hat{\psi}^n(i) + r_n(i, v_1, v_2) \hat{\psi}^n(i) \right] \\ &= \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_2} \hat{\psi}^n(i) + r_n(i, v_1, v_2) \hat{\psi}^n(i) \right] \\ (4.14) \quad &= \inf_{v_1 \in V_1} \left[\Pi_{v_1, v_{2n}^*(i)} \hat{\psi}^n(i) + r_n(i, v_1, v_{2n}^*(i)) \hat{\psi}^n(i) \right] \end{aligned}$$

For $v_n := (v_1, v_{2n}^*) \in \mathcal{M}_1 \times \mathcal{M}_2$, let $Y(\cdot)$ be the process (4.8) corresponding to v_n with initial condition $i \in S$. Then using Itô-Dynkin's formula and (4.14), we get

$$E_i^{v_1, v_{2n}^*} \left[e^{\int_0^T (r_n(Y(s), v_1(s, Y(s-)), v_{2n}^*(s, Y(s-))) - \rho_n) ds} \hat{\psi}^n(Y(T)) \right] - \hat{\psi}^n(i) \geq 0.$$

From Remark 4.1 it follows that for each n , $\hat{\psi}^n$ is bounded. Therefore we have

$$\hat{\psi}^n(i) \leq K(n) E_i^{v_1, v_{2n}^*} \left[e^{\int_0^T (r_n(Y(s), v_1(s, Y(s-)), v_{2n}^*(s, Y(s-))) - \rho_n) ds} \right].$$

Taking logarithm, dividing by T and by letting $T \rightarrow \infty$, we get

$$\rho_n \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \ln E_i^{v_1, v_{2n}^*} \left[e^{\int_0^T r_n(Y(s), v_1(s, Y(s-)), v_{2n}^*(s, Y(s-))) ds} \right].$$

Since $v_1 \in \mathcal{M}_1$ is arbitrary, it follows that

$$\rho_n \leq \inf_{v_1 \in \mathcal{M}_1} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln E_i^{v_1, v_{2n}^*} \left[e^{\int_0^T r_n(Y(s), v_1(s, Y(s-)), v_{2n}^*(s, Y(s-))) ds} \right].$$

Therefore we have

$$\rho_n \leq \sup_{v_2 \in \mathcal{M}_2} \inf_{v_1 \in \mathcal{M}_1} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln E_i^{v_1, v_2} \left[e^{\int_0^T r_n(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right].$$

Since $r_n \leq r$, we have

$$\rho_n \leq \sup_{v_2 \in \mathcal{M}_2} \inf_{v_1 \in \mathcal{M}_1} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln E_i^{v_1, v_2} \left[e^{\int_0^T r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right].$$

□

Now by using Theorem 4.2, one can closely mimic the arguments in the proof of [[8], Theorem 3.3] to prove the following.

Theorem 4.3. *Assume (A1) and $\|r\|_\infty \leq \delta$, where $\delta > 0$ is given in (A1). Then the equation (4.3) has a solution $(\rho, \hat{\psi}(i))$ satisfying $\hat{\psi}(i_0) = 1$. Also*

$$\rho \leq \sup_{v_2 \in \mathcal{M}_2} \inf_{v_1 \in \mathcal{M}_1} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln E_i^{v_1, v_2} \left[e^{\int_0^T r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right].$$

To prove that the ρ in Theorem 4.3 is indeed the value of the game. We used the atomic structure of the state dynamics, as in [8]. Let $v_1^* \in \mathcal{S}_1$ be the outer minimizing selector in (4.3) for player 1, $v_2 \in \mathcal{M}_2$ any strategy for player 2 and let $Y(\cdot)$ be a continuous time Markov chain corresponding to $(v_1^*, v_2) \in \mathcal{S}_1 \times \mathcal{M}_2$.

Define the twisted kernel $\tilde{P}(j, i)$ associated with $Y(\cdot)$ and r as follows.

$$(4.16) \quad \sum_{j \in S} h(j) \tilde{P}(j, i) = \frac{E_i^{v_1^*, v_2} [e^{\int_0^1 r(Y(s), v_1^*(Y(s-)), v_2(s, Y(s-))) ds} h(Y(1))]}{E_i^{v_1^*, v_2} [e^{\int_0^1 r(Y(s), v_1^*(Y(s-)), v_2(s, Y(s-))) ds}]}, i \in S, h \in B(S).$$

Set

$$e^{\hat{r}(i)} = E_i^{v_1^*, v_2} [e^{\int_0^1 r(Y(s), v_1^*(Y(s-)), v_2(s, Y(s-))) ds}].$$

Let $\{\tilde{Y}_n\}$ be a Markov chain on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ with transition kernel $\tilde{P}(\cdot, \cdot)$ and initial condition $i \in S$. We will denote the corresponding expectation by $\tilde{E}_i[\cdot]$.

Fix $i \in S$. Let $\{\tilde{Y}_n\}$ be the Markov chain given by (4.16) with $\tilde{Y}_0 = i$. Set $\tilde{\tau} = \inf\{n \geq 1 | \tilde{Y}_n = i\} := \tilde{\tau}_1$ and $\tilde{\tau}_{n+1} = \inf\{n \geq \tilde{\tau}_n + 1 | \tilde{Y}_n = i\}$. Define

$$D(\rho) = \tilde{E}_i[e^{\sum_{n=1}^{\tilde{\tau}} (\hat{r}(\tilde{Y}_n) - \rho)}].$$

We state the following lemmas which play a crucial role in this section. Since the proofs of these results closely mimic the corresponding proofs in [8], we omit the details.

Lemma 4.5. *Assume (A1) and $\|r\|_\infty < \delta$. Then*

$$(4.17) \quad D(\rho) \leq 1.$$

Lemma 4.6. *Assume (A1). Then for each $i \in S$ such that $W(i) \geq 1 + \frac{be^{\frac{3\delta}{2}}}{e^{\frac{\delta}{2}} - 1}$,*

$$\tilde{E}_i[e^{\delta\tilde{\tau}/2}] \leq e^{-\delta/2}(W(i) + be^{3\delta/2}),$$

where $\delta > 0$ is given in (A1).

Define $C_0 = \{i \in S : W(i) \geq 1 + \frac{be^{\frac{3\delta}{2}}}{e^{\frac{\delta}{2}} - 1}\}$. Now we state and prove the main theorem.

Theorem 4.4. *Assume (A1) and $\|r\|_\infty < \frac{\delta}{2}$, where $\delta > 0$ is given in (A1). Let $(\rho, \hat{\psi}(i))$ be the solution obtained in Theorem 4.3. Then*

$$\begin{aligned} \rho &= \inf_{v_1(\cdot) \in \mathcal{M}_1} \sup_{v_2(\cdot) \in \mathcal{M}_2} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln E_i^{v_1, v_2} \left[e^{\int_0^T r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right] \\ &= \sup_{v_2(\cdot) \in \mathcal{M}_2} \inf_{v_1(\cdot) \in \mathcal{M}_1} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln E_i^{v_1, v_2} \left[e^{\int_0^T r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right], \end{aligned}$$

i.e., ρ is the value of the risk-sensitive ergodic game. Furthermore there exists a pair of saddle point stationary Markov strategies (v_1^*, v_2^*) such that v_1^* is the outer minimizing selector in (4.3), and v_2^* is the outer maximizing selector in (4.3).

Proof. In view of Theorem 4.3, it remains to show that

$$\rho \geq \inf_{v_1(\cdot) \in \mathcal{M}_1} \sup_{v_2(\cdot) \in \mathcal{M}_2} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln E_i^{v_1, v_2} \left[e^{\int_0^T r(Y(s), v_1(s, Y(s-)), v_2(s, Y(s-))) ds} \right],$$

and the existence of a pair of saddle point stationary Markov strategies (v_1^*, v_2^*) such that v_1^* is the outer minimizing selector in (4.3), and v_2^* is the outer maximizing selector in (4.3).

Fix an $i \in C_0$ and $N \in \mathbb{N}$, define

$$e^{B_N(i)} = \tilde{E}_i \left[\exp \left\{ \sum_{k=1}^{N \wedge \tilde{\tau}} (\hat{r}(\tilde{Y}_k) - \rho) \right\} \right] \text{ for } N \in \mathbb{N}.$$

Arguing as in the proof of [[8], Theorem 3.5], it follows that

$$e^{B_{N+k}(i)} \geq \tilde{E}_i \left[\exp \left\{ \sum_{m=1}^k (\hat{r}(\tilde{Y}_m) - \rho) - N(\|\hat{r}\|_\infty + \rho) \right\} \right].$$

From Lemma 4.6, it follows that for each $i \in C_0$, $e^{B_N(i)} \leq e^{-\delta/2}(W(i) + be^{3\delta/2})$. Therefore taking logarithm in both sides and letting $k \rightarrow \infty$ we get

$$\limsup_k \frac{1}{k} \ln \tilde{E}_i \left[\exp \left\{ \sum_{m=0}^{k-1} \hat{r}(\tilde{Y}_m) \right\} \right] \leq \rho.$$

By using mathematical induction, we show that $\forall k \in \mathbb{N}$

$$E_i^{v_1^*, v_2} \left[\exp \left\{ \int_0^k r(Y(s), v_1^*(Y(s-)), v_2(s, Y(s-))) ds \right\} \right] = \tilde{E}_i \left[\exp \left\{ \sum_{i=0}^{k-1} \hat{r}(\tilde{Y}_i) \right\} \right].$$

That is true for $k = 1$, by definition. Let this be true for $k = n$. Then for $k = n + 1$

$$\begin{aligned} & \tilde{E}_i \left[\exp \left\{ \sum_{i=0}^n \hat{r}(\tilde{Y}_i) \right\} \right] \\ &= \tilde{E}_i \left[\exp \{ \hat{r}(i) \} \tilde{E}_{\tilde{Y}_1} \left[\exp \left\{ \sum_{i=0}^{n-1} \hat{r}(\tilde{Y}_i) \right\} \right] \right] \\ &= \tilde{E}_i \left[\exp \{ \hat{r}(i) \} E_{\tilde{Y}_1} \left[\exp \left\{ \int_0^n r(Y(s), v_1^*(Y(s-)), v_2(s, Y(s-))) ds \right\} \right] \right] \\ &= E_i^{v_1^*, v_2} \left[\exp \left\{ \int_0^1 r(Y(s), v_1^*(Y(s-)), v_2(s, Y(s-))) ds \right\} \right. \\ & \quad \left. E_{Y(1)}^{v_1^*, v_2} \left[\exp \left\{ \int_0^n r(Y(s), v_1^*(Y(s-)), v_2(s, Y(s-))) ds \right\} \right] \right] \\ &= E_i^{v_1^*, v_2} \left[\exp \left\{ \int_0^{n+1} r(Y(s), v_1^*(Y(s-)), v_2(s, Y(s-))) ds \right\} \right]. \end{aligned}$$

Hence we get

$$\rho \geq \limsup_{T \rightarrow \infty} \frac{1}{T} \ln E_i^{v_1^*, v_2} \left[e^{\int_0^T r(Y(s), v_1^*(Y(s-)), v_2(s, Y(s-))) ds} \right], i \in C_0.$$

Since $v_2 \in \mathcal{M}_2$ is arbitrary, it follows that

$$\rho \geq \sup_{v_2 \in \mathcal{M}_2} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln E_i^{v_1^*, v_2} \left[e^{\int_0^T r(Y(s), v_1^*(Y(s-)), v_2(s, Y(s-))) ds} \right], i \in C_0.$$

Therefore we have

$$\rho \geq \inf_{v_1 \in \mathcal{M}_1} \sup_{v_2 \in \mathcal{M}_2} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln E_i^{v_1^*, v_2} \left[e^{\int_0^T r(Y(s), v_1^*(Y(s-)), v_2(s, Y(s-))) ds} \right], i \in C_0.$$

Thus

$$\rho = \limsup_{T \rightarrow \infty} \frac{1}{T} \ln E_i^{v_1^*, v_2^*} \left[e^{\int_0^T r(Y(s), v_1^*(Y(s-)), v_2^*(Y(s-))) ds} \right], i \in C_0.$$

Arguing as in the proof of [[8], Theorem 3.5], it follows that the above equation holds for all $i \in S$, i.e.,

$$\rho = \limsup_{T \rightarrow \infty} \frac{1}{T} \ln E_i^{v_1^*, v_2^*} \left[e^{\int_0^T r(Y(s), v_1^*(Y(s-)), v_2^*(Y(s-))) ds} \right], i \in S.$$

It is easy to check that ρ is the value of the risk-sensitive ergodic game. Moreover, the pair of stationary Markov strategies (v_1^*, v_2^*) where v_1^* is the outer minimizing selector in (4.3), v_2^* is the outer maximizing selector in (4.3) forms a saddle point equilibrium. This completes the proof. \square

5. CONCLUSIONS

We have studied zero-sum risk-sensitive stochastic games for continuous time Markov chains on a countable state space. For the ergodic case we have taken the risk sensitive parameter $\theta = 1$ for the sake of simplicity. If we choose any other $\theta \in (0, \Theta)$, then the assumption in Theorem 4.4 has to be modified to $\theta \|r\|_\infty < \frac{\delta}{2}$. This is the so called small cost criterion which is standard in the literature [1], [2], [5], [8]. The corresponding non-zero sum case is currently under investigation.

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